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Thus, a, b, c, and d are known. The right-hand member of (1) will be the difference of two squares, and may be factored if we put $X - b = (x + t)^2$; *i.e.*, $X = (x + t)^2 + b$.

In the example given, $p = q = 0$, $r = -2$. The value $c = -1$ gives $n = 1$; whence, $a = 3/2$, $d = 5/3$, $b = -17/4$. $X^3 - 2 = (X + \frac{17}{4})(X - 1)^2 - \frac{1}{4}(3X - 5)^2$. Letting $X + \frac{17}{4} = (x + \frac{1}{2})^2$; *i.e.*, $X = x^2 + x - 4$, and substituting, we have

$$\begin{aligned} X^3 - 2 &= [(x + \tfrac{1}{2})(x^2 + x - 5) + \tfrac{1}{2}(3x^2 + 3x - 17)][(x + \tfrac{1}{2})(x^2 + x - 5) - \tfrac{1}{2}(3x^2 + 3x - 17)] \\ &= (x^3 + 3x^2 - 3x - 11)(x^3 - 6x + 6). \end{aligned}$$

2885 [1921, 139].

If A, B, C, X, Y are given collinear points, construct Z so that $\{ABCX\} + \{ABCY\} = \{ABCZ\}$, where $\{ABCX\}$ denotes the cross-ratio of the points A, B, C, X. [From *the Math. Tripos Exam.*, Cambridge, Eng., 1905.]

I. Solution by Gertrude I. McCain, Westminster College, New Wilmington, Pa.

If $\{ABCX\} + \{ABCY\} = \{ABCZ\}$, then $\frac{AB}{BC} : \frac{AX}{XC} + \frac{AB}{BC} : \frac{AY}{YC} = \frac{AB}{BC} : \frac{AZ}{ZC} \cdot$ Dividing by $\frac{AB}{BC}$, $\frac{XC}{AX} + \frac{YC}{AY} = \frac{ZC}{AZ} \cdot$ But $XC = -CX = -(AX - AC)$, $YC = -CY = -(AY - AC)$, and $ZC = -CZ = -(AZ - AC)$. Making the substitutions and adding 2 to each side of the equation, we have $\frac{AC}{AX} + \frac{AC}{AY} = \frac{AC}{AZ} + 1$, or $\frac{1}{AX} + \frac{1}{AY} = \frac{1}{AZ} + \frac{1}{AC} \cdot$ Let $AX = \cot \theta$, $AY = \cot \varphi$, $AC = \cot \psi$, and $AZ = \cot \tau$. Considering AX, AY and AC as line functions, construct the angles θ, φ and ψ, using a circle with radius unity. Also construct the tangents of θ, φ, and ψ as line functions using the same circle.

It is now possible to lay off $\tan \theta$ and $\tan \varphi$ as lengths on a line and from their sum subtract $\tan \psi$.

But $\frac{1}{AX} + \frac{1}{AY} - \frac{1}{AC} = \frac{1}{AZ} = \tan \tau$. Cot τ or AZ may then be constructed as a line function, and the point Z located on the line with A, B, C, X, Y.

II. Solution by Otto Dunkel, Washington University.

After showing that $1/AX + 1/AY = 1/AZ + 1/AC$, it should be observed that this result shows that the two sets of points A, X, Y and A, Z, C have a common fourth harmonic point K. Hence K can be constructed from the first set and then Z as the harmonic conjugate of C with respect to A and K, by the quadrilateral construction, thus giving a non-metrical construction. This would then give a projective definition of the sum of two cross-ratios. This result may be obtained by use of abridged notation as follows:

If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\xi = 0$, $\eta = 0$ are the equations of the points A, B, C, X, Y, then the expressions to the left in these equations may be chosen so that

$$\beta = b\alpha + \gamma, \qquad \xi = x\alpha + \gamma, \qquad \eta = y\alpha + \gamma,$$

and then $\{ABCX\} = x/b$, $\{ABCY\} = y/b$. It follows that $(x - y)\alpha = \xi - \eta$, and if K is the harmonic conjugate of A with respect to X, Y, its equation may be written $(x - y)\kappa = \xi + \eta = 0$ or $(x - y)\kappa = (x + y)\alpha + 2\gamma$. Since $2\gamma = (x - y)\kappa - (x + y)\alpha$, Z, the harmonic conjugate of C with respect to K, A, may have its equation written $2\zeta = (x - y)\kappa + (x + y)\alpha = 0$ or $\zeta = (x + y)\alpha + \gamma$. Hence $\{ABCZ\} = (x + y)/b = \{ABCX\} + \{ABCY\}$.

Also solved by Nathan Altshiller-Court, J. W. Clawson, Arthur Pelletier, and F. L. Wilmer.

2887 [1921, 139]. Proposed by the late L. G. WELD.

A carpenter's square moves with its outer edges in contact with two round pegs of given equal diameters. Define the locus of the "heel" of the square.

Solution by J. B. Reynolds, Lehigh University.

Let the radius of either peg be r and the distance between their centers $\sqrt{2}a$. Let the axes be taken so that the centers of the circles are at the points $(a, 0)$ and $(0, a)$, and assume that the square is placed above and to the right of the two circles, respectively. Also let θ be the angle which the side tangent to the former circle makes with the x-axis. The limiting positions of the square will be when the heel is just in contact with one of the circular pegs; that is, for example, when $a \cos \theta = r + a \sin \theta$, or $\theta = \frac{\pi}{4} - \sin^{-1} \frac{r}{\sqrt{2}a}$, so that the physical limits of θ are $\pm \left(\frac{\pi}{4} - \sin^{-1} \frac{r}{\sqrt{2}a}\right)$.

The points of tangency for any position of the square will be $(a - r \sin \theta,\ r \cos \theta)$ and $(r \cos \theta,\ a + r \sin \theta)$ and the equations of these tangents will be, respectively,

$$-(x - a) \sin \theta + y \cos \theta = r \qquad \text{and} \qquad x \cos \theta + (y - a) \sin \theta = r,$$

from which we get

$$x = -r(\sin \theta - \cos \theta) + a \sin \theta(\sin \theta + \cos \theta),$$
$$y = r(\sin \theta + \cos \theta) + a \sin \theta(\sin \theta - \cos \theta)$$

for the parametric equations defining the curve, with $-\left(\frac{\pi}{4} - \sin^{-1} \frac{r}{\sqrt{2}a}\right) \leqq \theta \leqq \left(\frac{\pi}{4} - \sin^{-1} \frac{r}{\sqrt{2}a}\right)$ for that portion subject to the physical limitations of the problem. By elimination of θ, we get for the general equation of the curve[1]

$$2r^2[x^2 + y^2 - 2a(x + y) + 2a^2] = [x^2 + y^2 - ax - ay]^2.$$

If in this $r = 0$, we have the circle $x^2 + y^2 = ax + ay$, as we should. For the above limits of θ to be correct, the shorter arm (18 inches) of the square must be $> \sqrt{(2a^2 - r^2)} - r$.

2895 [1921, 184]. Proposed by R. M. MATHEWS, Wesleyan University.

To construct an equilateral triangle with its vertices lying on: (*a*) any three coplanar lines; (*b*) three parallels in space; and (*c*) any three lines in space.

SOLUTION BY F. L. WILMER, Omaha, Neb.

(*a*) Let the three lines be l_1, l_2 and l_3. When they are not parallel we may suppose that the first two intersect in O. We bisect one of the angles between them and let the bisector meet l_3 in C. Then draw two lines from C, making angles of 30 degrees with CO (or with OC produced) and cutting l_1 and l_2 in A and B, respectively. ABC will be an equilateral triangle and there will generally be four triangles that can be constructed in this way.

When the three lines are parallel, draw a secant cutting them in P_1, P_2, P_3, respectively, and construct an equilateral triangle with P_1P_3 as one side and Q, say, as the third vertex. Rotate this triangle about P_2 until Q falls on l_2 and let the triangle in the new position be $P_1'Q'P_3'$. If we denote by A and B the points where $P_1'P_3'$ cuts l_1 and l_3, respectively, we shall find that the equilateral triangle ABC, constructed on AB on the same side of AB as Q', will satisfy the conditions of the problem; for $P_2P_1'/P_2A = P_2P_3'/P_2B$. Therefore P_2 is a center of similitude for the triangles $P_1'Q'P_3'$ and ABC, and C lies on P_2Q' or l_2.

(*b*) Consider the prismatic surface with edges l_1, l_2, l_3 and suppose that an equilateral triangle of side L can be placed with its three vertices on these three lines, respectively. Let a, b, c be the lengths of the sides of a right section $a \leqq b \leqq c$. Then

$$\sqrt{L^2 - a^2} = \sqrt{L^2 - b^2} + \sqrt{L^2 - c^2} \tag{1}$$

or

$$3L^4 - 2(a^2 + b^2 + c^2)L^2 + 16A^2 = 0, \tag{2}$$

where A is the area of the right section.

If L^2 is replaced by c^2, the expression in the first member of (2) reduces to $-(b^2 - a^2)^2$ and this shows that the equation in L^2 has two positive roots, one less than c^2 and the other greater than c^2 (unless $a = b$). The root greater than c^2 makes the three terms of (1) all real and satisfies (1) as written, since the first term of (1) is greater than or equal to each of the others.

(*c*) Partial solution—Instead of the three given lines, we may consider two non-coplanar lines and a point. Take the common perpendicular of the two for the z-axis and the bisectors of the angles made by their projections on the mid-plane as the axes of x and y. Then we may let

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[1] If we take the line joining the centers of the circles as the x-axis, with the origin at the mid-point of that line, the equation will be $(x^2 + y^2 - b^2)^2 = 2r^2[x^2 + (y - b)^2]$, where $2b$ is the distance between the centers.—EDITORS.